# VEKTORANALYS

Kursvecka 5

# övningar

## PROBLEM 1

Use “nablaräkning” to verify: $rot\left(\phi\overline{A}\right) = grad\,\phi \times \overline{A} + \phi\, rot\overline{A}$ **ID3**

## SOLUTION

*Add the dots*

$$rot\left(\phi\overline{A}\right) = \nabla\times\left(\phi\overline{A}\right) = \nabla\times\left(\underset{\bullet}{\phi}\overline{A}\right) + \nabla\times\left(\phi\underset{\bullet}{\overline{A}}\right) =$$

*Now nabla can be considered as a vector:* $\overline{n}\times\left(c\overline{a}\right)+\overline{n}\times\left(c\overline{a}\right)=\left(\overline{n}c\right)\times\overline{a}+c\left(\overline{n}\times\overline{a}\right)$ *(because c is a scalar)*

$$= \left(\nabla\underset{\bullet}{\phi}\right)\times\overline{A} + \phi\left(\nabla\times\underset{\bullet}{\overline{A}}\right) =$$

$$= \left(\nabla\phi\right)\times\overline{A} + \phi\left(\nabla\times\overline{A}\right)$$

## PROBLEM 2

Use “nablaräkning” to verify: $div\, rot\overline{A} = 0$ **ID8**

## SOLUTION

*Now nabla can be considered as a vector.*

$$div\, rot\overline{A} = \nabla\cdot\left(\nabla\times\underset{\bullet}{\overline{A}}\right) =$$

Because: $\overline{n}\cdot\left(\overline{n}\times\overline{a}\right) = \overline{a}\cdot\left(\underbrace{\overline{n}\times\overline{n}}_{=0}\right)$

$$= \underset{\bullet}{\overline{A}}\cdot\left(\nabla\times\nabla\right) = \left(\nabla\times\nabla\right)\cdot\underset{\bullet}{\overline{A}} = 0$$

## PROBLEM 3

Use “nablaräkning” to verify: $\left(\overline{A}\times\nabla\right)\times\overline{A}=\frac{1}{2}\nabla A^2-\overline{A}\left(\nabla\cdot\overline{A}\right)$

SOLUTION

$$\left(\overline{A}\times\nabla\right)\times\overline{A}=\left(\overline{A}\times\nabla\right)\times\underset{\cdot}{\overline{A}}=$$

$$\left(\overline{a}\times\overline{n}\right)\times\overline{b}=\left(\overline{a}\cdot\overline{b}\right)\overline{n}-\left(\overline{n}\cdot\overline{b}\right)\overline{a}$$

$$=\nabla\left(\overline{A}\cdot\underset{\cdot}{\overline{A}}\right)-\overline{A}\left(\nabla\cdot\underset{\cdot}{\overline{A}}\right)=$$

$$\nabla A^2=\nabla\left(\overline{A}\cdot\overline{A}\right)=\nabla\left(\underset{\cdot}{\overline{A}}\cdot\overline{A}\right)+\nabla\left(\overline{A}\cdot\underset{\cdot}{\overline{A}}\right)=2\nabla\left(\overline{A}\cdot\underset{\cdot}{\overline{A}}\right)$$

$$\Rightarrow\quad\nabla\left(\overline{A}\cdot\underset{\cdot}{\overline{A}}\right)=\frac{1}{2}\nabla A^2$$

$$=\frac{1}{2}\nabla A^2-\overline{A}\left(\nabla\cdot\overline{A}\right)$$

## PROBLEM 4

Use “nablaräkning” to simplify: $\overline{B}=\overline{A}\times\left(\nabla\times\overline{A}\right)-\left(\overline{A}\times\nabla\right)\times\overline{A}$

SOLUTION

$$\overline{B}=\overline{A}\times\left(\nabla\times\overline{A}\right)-\left(\overline{A}\times\nabla\right)\times\overline{A}=\overline{A}\times\left(\nabla\times\underset{\cdot}{\overline{A}}\right)-\left(\overline{A}\times\nabla\right)\times\underset{\cdot}{\overline{A}}=\overline{A}\times\left(\nabla\times\underset{\cdot}{\overline{A}}\right)+\underset{\cdot}{\overline{A}}\times\left(\overline{A}\times\nabla\right)=$$

using $\overline{a}\times\left(\overline{b}\times\overline{c}\right)=\overline{b}\left(\overline{a}\cdot\overline{c}\right)-\overline{c}\left(\overline{a}\cdot\overline{b}\right)$

$$=\nabla\left(\overline{A}\cdot\underset{\cdot}{\overline{A}}\right)-\underset{\cdot}{\overline{A}}\left(\overline{A}\cdot\nabla\right)-\nabla\left(\underset{\cdot}{\overline{A}}\cdot\overline{A}\right)+\overline{A}\left(\underset{\cdot}{\overline{A}}\cdot\nabla\right)=$$

$$=\nabla\left(\underset{\cdot}{\overline{A}}\cdot\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\underset{\cdot}{\overline{A}}-\nabla\left(\underset{\cdot}{\overline{A}}\cdot\overline{A}\right)+\overline{A}\left(\nabla\cdot\underset{\cdot}{\overline{A}}\right)=$$

$$=-\left(\overline{A}\cdot\nabla\right)\underset{\cdot}{\overline{A}}+\overline{A}\left(\nabla\cdot\underset{\cdot}{\overline{A}}\right)=\overline{A}\left(\nabla\cdot\overline{A}\right)-\left(\overline{A}\cdot\nabla\right)\overline{A}$$

## PROBLEM 5

Calculate $\varepsilon_{ijk}\varepsilon_{ljk}$

## SOLUTION

We know that $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Therefore:

*We re-arrange the suffixes to have an expression similar to*

*the same expression with m=j*

$$\varepsilon_{ijk}\varepsilon_{ljk} = \varepsilon_{ijk}\varepsilon_{klj} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = \delta_{il}3 - \delta_{il} = 2\delta_{il}$$

*even permutations does NOT change the sign*

*Remember that* $\delta_{ii} = 3$

*Remember that* $\delta_{km}p_m = p_k$

**PROBLEM 6**

Prove $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$

using the suffix notation.

SOLUTION

We know that the *i-component* of the cross product can be written as: $\left(\bar{a}\times\bar{b}\right)_i=\varepsilon_{ijk}a_j b_k$
Therefore:

$$\left(\bar{b}\times\bar{c}\right)_k=\varepsilon_{klm}b_l c_m$$

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)_i=\varepsilon_{ijk}a_j\left(\bar{b}\times\bar{c}\right)_k=$$

$$\varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

$$=\varepsilon_{ijk}a_j\varepsilon_{klm}b_l c_m=$$

$$\delta_{ij}d_{kj}=d_{ki}$$

$$=\varepsilon_{ijk}\varepsilon_{klm}a_j b_l c_m=$$

$$=\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)a_j b_l c_m=\delta_{il}\delta_{jm}a_j b_l c_m-\delta_{im}\delta_{jl}a_j b_l c_m=$$

$$=a_m b_i c_m-a_l b_l c_i=\left(\bar{a}\cdot\bar{c}\right)b_i-\left(\bar{a}\cdot\bar{b}\right)c_i$$

**PROBLEM 7**

Use “indexräkning” to verify: $div\left(\overline{A}\times\overline{B}\right)=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$ **ID4**

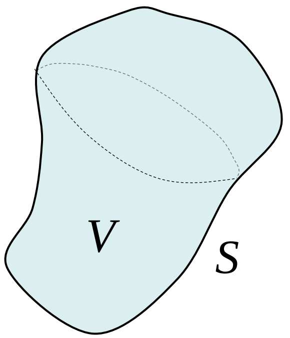
SOLUTION

$$div\left(\overline{A}\times\overline{B}\right)=\left(\overline{A}\times\overline{B}\right)_{,i}=\left(\varepsilon_{ijk}A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_jB_k\right)_{,i}=\varepsilon_{ijk}\left(A_{j,i}B_k+A_jB_{k.i}\right)=$$

$$=\varepsilon_{ijk}A_{j,i}B_k+\varepsilon_{ijk}A_jB_{k.i}=\varepsilon_{kij}A_{j,i}B_k-\varepsilon_{jik}B_{k.i}A_j=\left(rot\overline{A}\right)\cdot\overline{B}-\left(rot\overline{B}\right)\cdot\overline{A}$$

*remember that:* $\left(\nabla\times\overline{A}\right)_i=\varepsilon_{ijk}A_{k,j}$

## PROBLEM 8




Show that: $$\iiint_V \overline{r} \times rot\overline{A}\,dV = 2\iiint_V \overline{A}\,dV$$

if on the boundary surface $S$ of $V$ the vector field is $\overline{A} = 0$

## SOLUTION

Let's consider only the *i-th* component of the left hand side:

$$\hat{e}_i \cdot \iiint_V \overline{r} \times rot\overline{A}\,dV = \iiint_V \hat{e}_i \cdot \left(\overline{r} \times rot\overline{A}\right) dV = \iiint_V \left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) dV =$$

$$\overline{a} \cdot \left(\overline{b} \times \overline{c}\right) = \overline{c} \cdot \left(\overline{a} \times \overline{b}\right)$$

To continue, we must remember that: $\nabla \cdot \left(\overline{a} \times \overline{b}\right) = \overline{b} \cdot \left(\nabla \times \overline{a}\right) - \overline{a} \cdot \left(\nabla \times \overline{b}\right)$ (**ID4**)

therefore, $$\nabla \cdot \left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) = \left(\hat{e}_i \times \overline{r}\right) \cdot \left(\nabla \times \overline{A}\right) - \overline{A} \cdot \nabla \times \left(\hat{e}_i \times \overline{r}\right)$$

re-arranging the terms: $$\left(rot\overline{A}\right) \cdot \left(\hat{e}_i \times \overline{r}\right) = div\left(\overline{A} \times \left(\hat{e}_i \times \overline{r}\right)\right) + \overline{A} \cdot rot\left(\hat{e}_i \times \overline{r}\right)$$

*and we substitute*

$$= \iiint_V \left[ div\left(\overline{A}\times\left(\hat{e}_i \times \overline{r}\right)\right) + \overline{A}\cdot rot\left(\hat{e}_i \times \overline{r}\right) \right] dV =$$

$$= \iiint_V div\left(\overline{A}\times\left(\hat{e}_i \times \overline{r}\right)\right) dV + \iiint_V \overline{A}\cdot rot\left(\hat{e}_i \times \overline{r}\right) dV =$$

*Generalized Gauss theorem*

**ID5**

$$rot\left(\hat{e}_i \times \overline{r}\right) = \left(\overline{r}\cdot\nabla\right)\hat{e}_i - \left(\hat{e}_i\cdot\nabla\right)\overline{r} + \hat{e}_i\left(\nabla\cdot\overline{r}\right) - \overline{r}\left(\nabla\cdot\hat{e}_i\right)$$

$$= \quad 0 \quad - \frac{\partial\overline{r}}{\partial x_i} \quad + \quad 3\hat{e}_i \quad - \quad 0 \quad = 2\hat{e}_i$$

$$= \underbrace{\iint_S \left(\overline{A}\times\left(\hat{e}_i \times \overline{r}\right)\right)\cdot d\overline{S}}_{=0} + \iiint_V \overline{A}\cdot 2\hat{e}_i \, dV = 2\iiint_V A_i \, dV$$

Because on $S$, $\overline{A}=0$

So, we have: $$\hat{e}_i \cdot \iiint_V \overline{r}\times rot\overline{A} \, dV = 2\iiint_V A_i \, dV$$